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Letter to the Editor

Free vibration analysis of an unsymmetric trapezoidal membrane

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1. Introduction

We introduced an analytical approach for free vibration analysis of a composite rectangular membrane, composed of two homogeneous regions whose interface is oblique [1]. In the previous research, a special approach for solving the compatibility condition given along the *oblique* interface was presented. The main idea of the special approach was to define a local co-ordinate system, (x, y), along the oblique interface so that the x-axis overlaps with the interface. By the use of the local co-ordinate system, only the single geometric variable x was involved in the compatibility condition, because all points on the interface satisfy $y = 0$. Similarly in this paper dealing with a trapezoidal membrane, local co-ordinate systems are employed to simplify the fixed boundary conditions for two *oblique* edges among four edges of a trapezoidal membrane.

Although numerical methods such as the finite element method (FEM) [2,3] and the boundary element method (BEM) [4,5] can be used to find the eigenvalues and eigenvectors of an unsymmetric trapezoidal membrane, these methods need a great number of elements to obtain accurate results and, thus, a large amount of numerical calculation. Note that FEM and BEM need many elements to obtain fully converged results. To overcome the weak point of these methods, many analytical methods suitable for the geometric feature of the trapezoidal membrane have been proposed. In particular, there have been numerous papers on circular, annular, triangular, rhombic and parallelogram membranes (or plates). However, the literature reveals that *unsymmetric* trapezoidal membranes (or plates) have rarely been studied (Note that several papers have been published for *symmetric* trapezoidal membranes [6–8]). To the authors' best knowledge, only Chopra and Durvasula [9] proposed an analytical method for the free vibration analysis of simply supported unsymmetric trapezoidal plates. They basically used the Galerkin method in which an assumed solution satisfies all the boundary conditions (in general, the assumed solution

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does not satisfy the governing differential equation) [10]. Unlike in the Chopra and Durvasula' case, an assumed solution used in this paper satisfies the governing differential equation but does not satisfy all the boundary conditions. Our assumed solution does not satisfy the fixed boundary conditions given at the two *oblique* edges of a trapezoidal membrane.

In the paper, the sum of the eigensolutions of two semi-infinite membranes, which have infinite regions in the left and right directions, respectively, as shown in Fig. 2, was employed as the assumed solution of a trapezoidal membrane. To obtain the system matrix of which the singular values partially correspond to the eigenvalues of the membrane, the fixed boundary conditions given at the two oblique edges was applied to the assumed solution. Furthermore, a practical and intuitive way to remove spurious eigenvalues from the singular values of the system matrix was developed. Finally, the validity and accuracy of the eigenvalues and mode shapes found by the proposed method was verified by a comparison test.

2. Theoretical formulation

2.1. Approximate solution of a trapezoidal membrane

A trapezoidal membrane is represented in Fig. 1, where the shape of the membrane is characterized by base length a, height b, and skew angles (α_1 and α_2). An assumed solution $W(X, Y)$ for the free transverse vibration of the trapezoidal membrane may be expressed as the sum of two eigensolutions, i.e.,

$$W(X, Y) = W_{\mathrm{I}}(X, Y) + W_{\mathrm{II}}(X, Y), \tag{1}$$

where $W_{\mathrm{I}}(X, Y)$ and $W_{\mathrm{II}}(X, Y)$ represent the eigensolutions of two semi-infinite membranes illustrated in Fig. 2, respectively. Note that the first semi-infinite membrane is fixed at $X = 0$, $Y = 0$ and $Y = b$, and the second one at $X = a$, $Y = 0$ and $Y = b$.

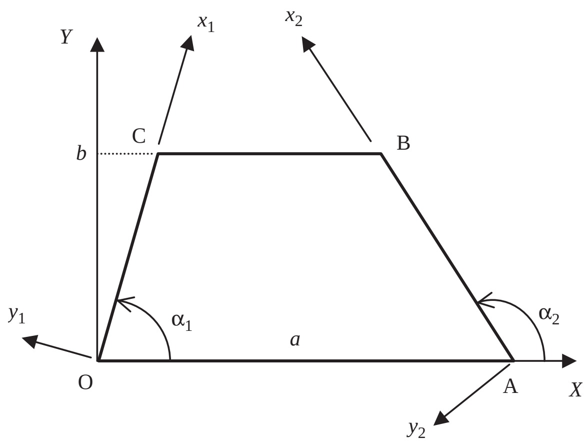

Fig. 1. General trapezoidal membrane of base length a, height b and skew angles (α_1 and α_2).

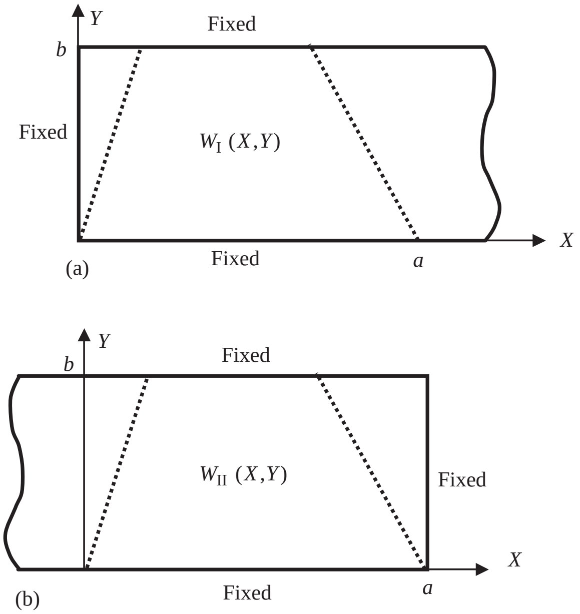

Fig. 2. Two semi-infinite membranes with fixed edges at (a) $X = 0$, $Y = 0$ and $Y = b$; (b) $X = a$, $Y = 0$ and $Y = b$.

Next, $W_{\mathrm{I}}(X, Y)$ and $W_{\mathrm{II}}(X, Y)$ are assumed as

$$W_{\mathrm{I}}(X, Y) = f_{\mathrm{I}}(X)g_{\mathrm{I}}(Y), \quad W_{\mathrm{II}}(X, Y) = f_{\mathrm{II}}(X)g_{\mathrm{II}}(Y). \tag{2, 3}$$

If Eqs. (2) and (3) are substituted into the governing differential equation of the membrane ($\nabla^2 W + k^2 W = 0$) and the fixed boundary conditions for each semi-infinite membrane are considered, one can obtain

$$W_{\mathrm{I}}(X, Y) = \sum_{m=1}^{N} A_m^{(\mathrm{I})} \sin k_x^{(\mathrm{I})} X \sin m\pi Y/b, \tag{4}$$

$$W_{\mathrm{II}}(X, Y) = \sum_{n=1}^{N} A_n^{(\mathrm{II})} \sin k_x^{(\mathrm{II})}(a - X) \sin n\pi Y/b, \tag{5}$$

where $k_x^{(\mathrm{I})} = \sqrt{k^2 - (m\pi/b)^2}$ and $k_x^{(\mathrm{II})} = \sqrt{k^2 - (n\pi/b)^2}$ (k denotes the frequency parameter).

2.2. Applying boundary conditions

The assumed solution $W(X, Y)$ satisfies not only the governing differential equation but also the fixed boundary conditions given along the two horizontal edges ($\overline{OA}$ and $\overline{BC}$ in Fig. 1) of the membrane, because both semi-infinite membranes are fixed at $Y = 0$ and $Y = b$ as illustrated in Fig. 2. However, $W(X, Y)$ does not satisfy the fixed boundary conditions given at the two oblique edges ($\overline{OC}$ and $\overline{AB}$ in Fig. 1) of the membrane. For $W(X, Y)$ to become an approximate solution for the free vibration of the trapezoidal membrane, $W(X, Y)$ should satisfy the fixed boundary conditions given at the two oblique edges. Therefore, a co-ordinate transformation was applied so that the fixed boundary conditions at the two oblique edges could be considered. Two local co-ordinate systems (x_1, y_1) and (x_2, y_2) shown in Fig. 1 were employed. The relationship between local co-ordinate system (x_i, y_i) and global co-ordinate system (X, Y) is given by

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{bmatrix} p_i & -q_i \\ q_i & p_i \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} + \begin{Bmatrix} X_i \\ 0 \end{Bmatrix}, \quad i = 1 \text{ or } 2, \tag{6}$$

where $p_i = \cos\alpha_i$, $q_i = \sin\alpha_i$, $X_1 = 0$ and $X_2 = a$.

2.2.1. Fixed boundary condition at oblique edge $\overline{OC}$

By the use of Eq. (6) with $i = 1$, the assumed solution $W(X, Y)$ can be expressed in terms of local co-ordinate system (x_1, y_1) as follows:

$$\begin{aligned} W(x_1, y_1) = &\sum_{m=1}^{N} A_m^{(\mathrm{I})} \sin k_x^{(\mathrm{I})}(p_1 x_1 - q_1 y_1) \sin m\pi(q_1 x_1 + p_1 y_1)/b \\ &+ \sum_{n=1}^{N} A_n^{(\mathrm{II})} \sin k_x^{(\mathrm{II})}(a - p_1 x_1 + q_1 y_1) \sin n\pi(q_1 x_1 + p_1 y_1)/b. \end{aligned} \tag{7}$$

Also, the fixed boundary condition given at oblique edge $\overline{OC}$ may be expressed as

$$W(x_1, y_1 = 0) = 0. \tag{8}$$

Applying Eq. (8) to Eq. (7) yields

$$\sum_{m=1}^{N} A_m^{(\mathrm{I})} \sin(k_x^{(\mathrm{I})} p_1 x_1) \sin(m\pi q_1 x_1/b) + \sum_{n=1}^{N} A_n^{(\mathrm{II})} \sin k_x^{(\mathrm{II})}(a - p_1 x_1) \sin(n\pi q_1 x_1/b) = 0. \tag{9}$$

Although the fixed boundary condition has been considered, geometric variable x_1 is still involved in Eq. (9). To remove x_1, the s-basis $\sin s\pi x_1/L_1$ (L_1 is the length of $\overline{OC}$) is multiplied to Eq. (9) under the assumption that the oblique edge harmonically vibrates with end points O and C fixed, and an integration procedure is performed from $x_1 = 0$ to $x_1 = L_1$. Then, Eq. (9) leads to

$$\sum_{m=1}^{N} SM_{sm}^{(\mathrm{I},1)} A_m^{(\mathrm{I})} + \sum_{n=1}^{N} SM_{sn}^{(\mathrm{II},1)} A_n^{(\mathrm{II})} = 0, \quad s = 1, 2, \ldots, N, \tag{10}$$

where $SM_{sm}^{(\mathrm{I},1)}$ and $SM_{sn}^{(\mathrm{II},1)}$ are given by

$$SM_{sm}^{(\mathrm{I},1)} = \int_0^{L_1} \sin k_x^{(\mathrm{I})} p_1 x_1 \sin(m\pi q_1 x_1/b) \sin(s\pi x_1/L_1)\, \mathrm{d}x_1, \tag{11}$$

$$SM_{sn}^{(\mathrm{II},1)} = \int_0^{L_1} \sin k_x^{(\mathrm{II})}(a - p_1 x_1) \sin(n\pi q_1 x_1/b) \sin(s\pi x_1/L_1)\, \mathrm{d}x_1. \tag{12}$$

For simplicity, Eq. (10) is rewritten in the matrix equation form

$$\mathbf{SM}^{(\mathrm{I},1)}\mathbf{A}^{(\mathrm{I})} + \mathbf{SM}^{(\mathrm{II},1)}\mathbf{A}^{(\mathrm{II})} = \mathbf{0}. \tag{13}$$

2.2.2. Fixed boundary condition at oblique edge $\overline{AB}$

Using Eq. (6) with $i = 2$, the fixed boundary condition at oblique edge $\overline{AB}$ can be written as

$$W(x_2, y_2 = 0) = 0, \tag{14}$$

and the assumed solution $W(X, Y)$ can be expressed as

$$\begin{aligned} W(x_2, y_2) = &\sum_{m=1}^{N} A_m^{(\mathrm{I})} \sin k_x^{(\mathrm{I})}(p_2x_2 - q_2y_2 + a)\sin m\pi(q_2x_2 + p_2y_2)/b \\ &+ \sum_{n=1}^{N} A_n^{(\mathrm{II})} \sin k_x^{(\mathrm{II})}(-p_2x_2 + q_2y_2)\sin n\pi(q_2x_2 + p_2y_2)/b. \end{aligned} \tag{15}$$

Applying Eq. (14) to Eq. (15) gives

$$\sum_{m=1}^{N} A_m^{(\mathrm{I})} \sin k_x^{(\mathrm{I})}(p_2x_2 + a)\sin(m\pi q_2x_2/b) + \sum_{n=1}^{N} A_n^{(\mathrm{II})} \sin k_x^{(\mathrm{II})}(-p_2x_2)\sin(n\pi q_2x_2/b) = 0. \tag{16}$$

As in Section 2.2.1, $\sin s\pi x_2/L_2$ (L_2 is the length of $\overline{AB}$) is multiplied to both sides of Eq. (16) and is integrated from $x_2 = 0$ to $x_2 = L_2$. Then, one can obtain

$$\sum_{m=1}^{N} SM_{sm}^{(\mathrm{I},2)} A_m^{(\mathrm{I})} + \sum_{n=1}^{N} SM_{sn}^{(\mathrm{II},2)} A_n^{(\mathrm{II})} = 0, \quad s = 1, 2, \ldots N, \tag{17}$$

where $SM_{sm}^{(\mathrm{I},2)}$ and $SM_{sn}^{(\mathrm{II},2)}$ are given by

$$SM_{sm}^{(\mathrm{I},2)} = \int_0^{L_2} \sin k_x^{(\mathrm{I})}(p_2x_2 + a)\sin(m\pi q_2x_2/b)\sin(s\pi x_2/L_2)\,\mathrm{d}x_2, \tag{18}$$

$$SM_{sn}^{(\mathrm{II},2)} = \int_0^{L_2} \sin k_x^{(\mathrm{II})}(-p_2x_2)\sin(n\pi q_2x_2/b)\sin(s\pi x_2/L_2)\,\mathrm{d}x_2. \tag{19}$$

Finally, Eqs. (17) is simplified in the matrix equation form

$$\mathbf{SM}^{(\mathrm{I},2)}\mathbf{A}^{(\mathrm{I})} + \mathbf{SM}^{(\mathrm{II},2)}\mathbf{A}^{(\mathrm{II})} = \mathbf{0}. \tag{20}$$

2.2.3. System matrix of a trapezoidal membrane

Eqs. (13) and (20) may be written in the single matrix equation

$$\mathbf{SM}_{2N}(k)\mathbf{A} = \mathbf{0}, \tag{21}$$

where square matrix $\mathbf{SM}_{2N}$ of order $2N$ is termed the *system matrix*; the system matrix and the unknown coefficient vector are given by

$$\mathbf{SM}_{2N} = \begin{bmatrix} \mathbf{SM}^{(\mathrm{I},1)} & \mathbf{SM}^{(\mathrm{II},1)} \\ \mathbf{SM}^{(\mathrm{I},2)} & \mathbf{SM}^{(\mathrm{II},2)} \end{bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} \mathbf{A}^{(\mathrm{I})} \\ \mathbf{A}^{(\mathrm{II})} \end{Bmatrix}. \tag{22, 23}$$

Note that $\mathbf{SM}_{2N}$ is a function of frequency parameter k because the frequency parameter is involved in the 4 sub-matrices of $\mathbf{SM}_{2N}$. From the fact that $\mathbf{A} = \mathbf{0}$ unless $\det[\mathbf{SM}_{2N}(k)] = 0$, the eigenvalues of a trapezoidal membrane can be calculated from

$$\det[\mathbf{SM}_{2N}(k)] = 0. \tag{24}$$

Furthermore, the mode shape of the ith eigenmode for the ith eigenvalue (Λ_i) can be obtained by plotting Eqs. (1), which is given by Eqs. (4) and (5). Note that unknown coefficients involved in Eqs. (4) and (5) can be replaced by the ith eigenvector satisfying Eq. (21) for $k = \Lambda_i$.

3. Numerical test and discussion

3.1. Singular values of the system matrix

In the numerical test, the proposed method calculated the eigenvalues and mode shapes of an unsymmetric trapezoidal membrane with $a = 2\,\mathrm{m}$, $b = 1\,\mathrm{m}$, $\alpha_1 = 60^\circ$ and $\alpha_2 = 110^\circ$. To find the roots of Eq. (24), logarithm values of $\det[\mathbf{SM}_{2N}(k)]$ were plotted in the frequency parameter range of $k = 3-10$ (see Fig. 3). From the fact that the fundamental eigenvalue of the trapezoidal membrane is larger that that of the rectangular membrane with $a = 2\,\mathrm{m}$ and $b = 1\,\mathrm{m}$, the minimum value of the sweeping range ($k = 3$–10) was determined as 3.

In the determinant curve of Fig. 3, frequency parameter values corresponding to 21 troughs represent the singular values of $\mathbf{SM}_{2N}(k)$, which are summarized in Table 1. The interesting fact that all the singular values of $\mathbf{SM}_{2N}(k)$ did not correspond to the eigenvalues of the trapezoidal membrane will be confirmed in the numerical test. The singular values corresponding to the eigenvalues of the trapezoidal membrane are termed the *correct singular values* and the other singular values are termed the *incorrect singular values* in the paper.

In Table 1, the singular values of $\mathbf{SM}_{2N}(k)$ are compared with the eigenvalues of the trapezoidal membrane calculated by FEM (ANSYS). This comparison reveals that only 8 singular values

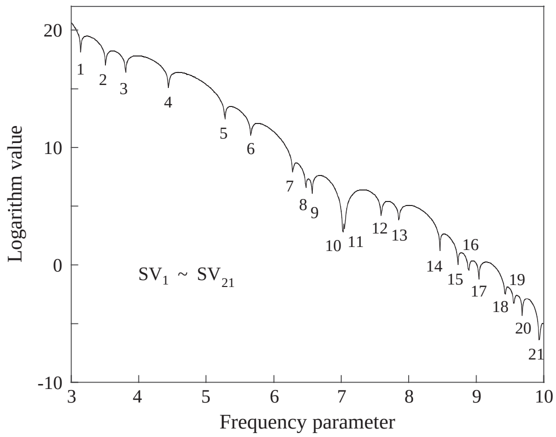

Fig. 3. Logarithm values of $\det[\mathbf{SM}_{2N}(k)]$ for the trapezoidal membrane ($N = 3$).

Table 1
Comparison between the singular values of $\mathbf{SM}_{2N}(k)$ obtained by the proposed method and the eigenvalues of the trapezoidal membrane computed by FEM (ANSYS)

No.	Trapezoidal membrane		Rectangular membrane		Cut-off frequency values
	Singular values of $\mathbf{SM}_{2N}(k)$	Eigenvalues by FEM (607 nodes)	Singular values of $\mathbf{SM}^{rec}_{2N}(k)$	Exact eigenvalues	
1	3.14		3.14		3.14
2	3.51		3.51	3.51	
3	3.81	3.82			
4	4.44		4.44	4.44	
5	5.28	5.29			
6	5.66		5.66	5.66	
7	6.28		6.28		6.28
8	6.48		6.48	6.48	
9	6.57	6.60			
10	7.03		7.03	7.03	
11	7.05	7.09			
12	7.59	7.64			
13	7.85		7.85	7.85	
14	8.46		8.46	8.46	
15	8.73	8.79			
16	8.89		8.89	8.89	
17	9.04	9.10			
18	9.42		9.42		9.42
19	9.56		9.56	9.56	
20	9.68	9.83			
21	9.93		9.93	9.93	

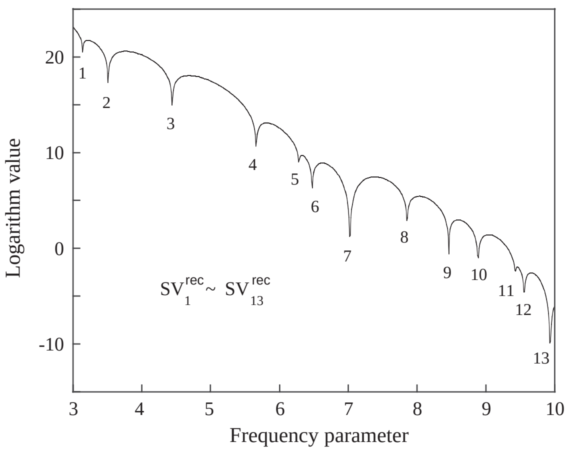

Fig. 4. Logarithm values of det[$\mathbf{SM}^{rec}_{2N}(k)$] for the rectangular membrane ($N = 3$).

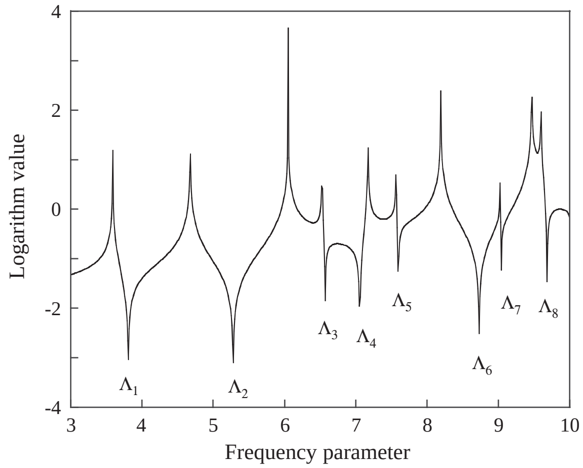

Fig. 5. Logarithm values of $\det[\mathbf{SM}_{2N}(k)]/\det[\mathbf{SM}_{2N}^{\text{rec}}(k)]$ for the trapezoidal membrane ($N = 3$).

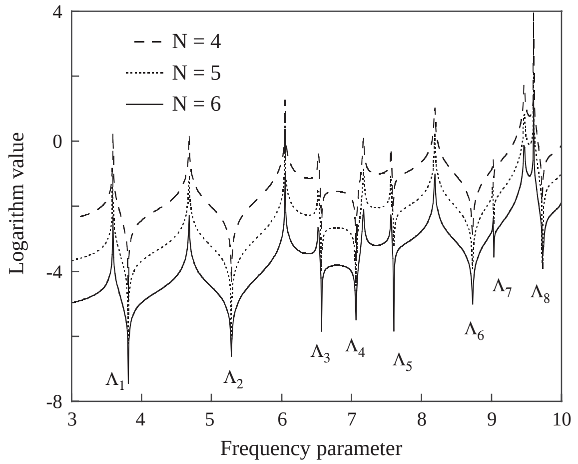

Fig. 6. Logarithm values of $\det[\mathbf{SM}_{2N}(k)]/\det[\mathbf{SM}_{2N}^{rec}(k)]$ for the trapezoidal membrane ($N = 4$, 5 and 6).

among 21 singular values coincide with the eigenvalues of the trapezoidal membrane. Note that the singular values of $\mathbf{SM}_{2N}(k)$ consist of 8 *correct singular values* and 13 *incorrect singular values*.

3.2. Removing the incorrect singular values

In the section, an intuitive way for removing the incorrect singular values is devised. First, the proposed method is employed to find the eigenvalues of the rectangular membrane with $a = 2\,\text{m}$ and $b = 1\,\text{m}$, $\alpha_1 = 0^\circ$ and $\alpha_2 = 0^\circ$ in Fig. 1. Then, the eigenvalues of the rectangular membrane may be obtained from the singular values of the system matrix $\mathbf{SM}_{2N}^{rec}(k)$ of the rectangular

Table 2
Convergence of eigenvalues of the trapezoidal membrane obtained by the proposed method and FEM (ANSYS)

	Present $N=3$	Present $N=4$	Present $N=5$	Present $N=6$	FEM $n_{nd}=1447$	FEM $n_{nd}=859$	FEM $n_{nd}=607$	FEM $n_{\mathrm{nd}}=313$
Λ_1	3.81	3.81	3.81	3.81	3.81	3.81	3.82	3.82
Λ_2	5.28	5.28	5.28	5.28	5.29	5.29	5.29	5.31
Λ_3	6.57	6.57	6.57	6.57	6.58	6.59	6.60	6.64
Λ_4	7.05	7.06	7.06	7.06	7.07	7.08	7.09	7.13
Λ_5	7.59	7.59	7.60	7.60	7.62	7.63	7.64	7.68
Λ_6	8.73	8.73	8.73	8.73	8.75	8.77	8.79	8.85
Λ_7	9.04	9.03	9.03	9.03	9.06	9.07	9.10	9.16
Λ_8	9.68	9.72	9.73	9.73	9.78	9.81	9.83	10.02

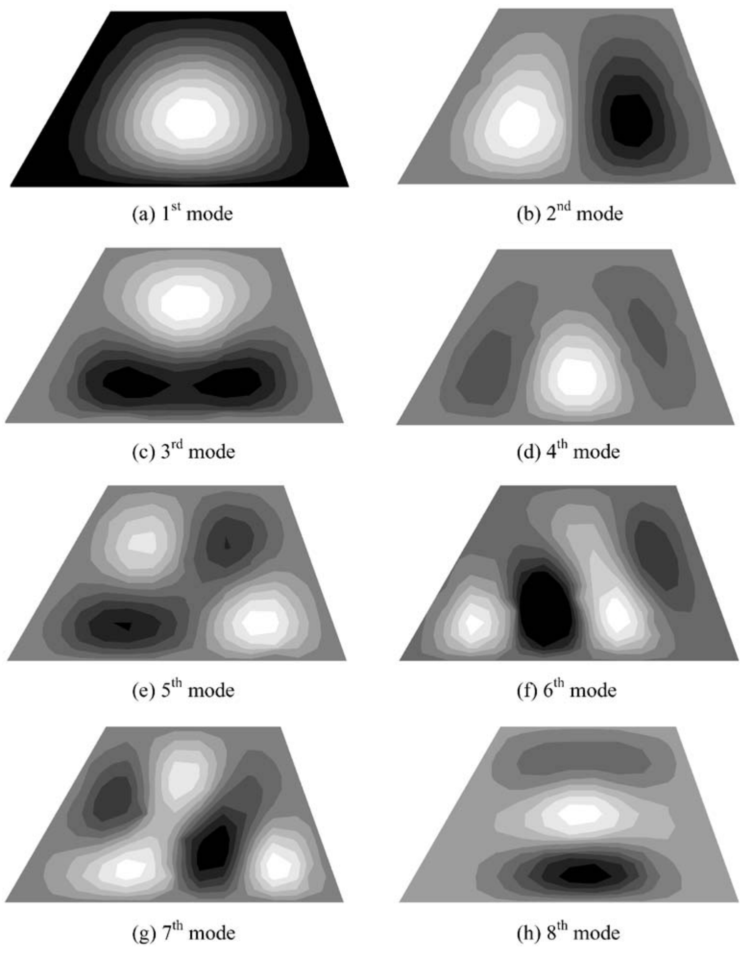

Fig. 7. First eight mode shapes of the trapezoidal membrane by the proposed method for $N = 4$.

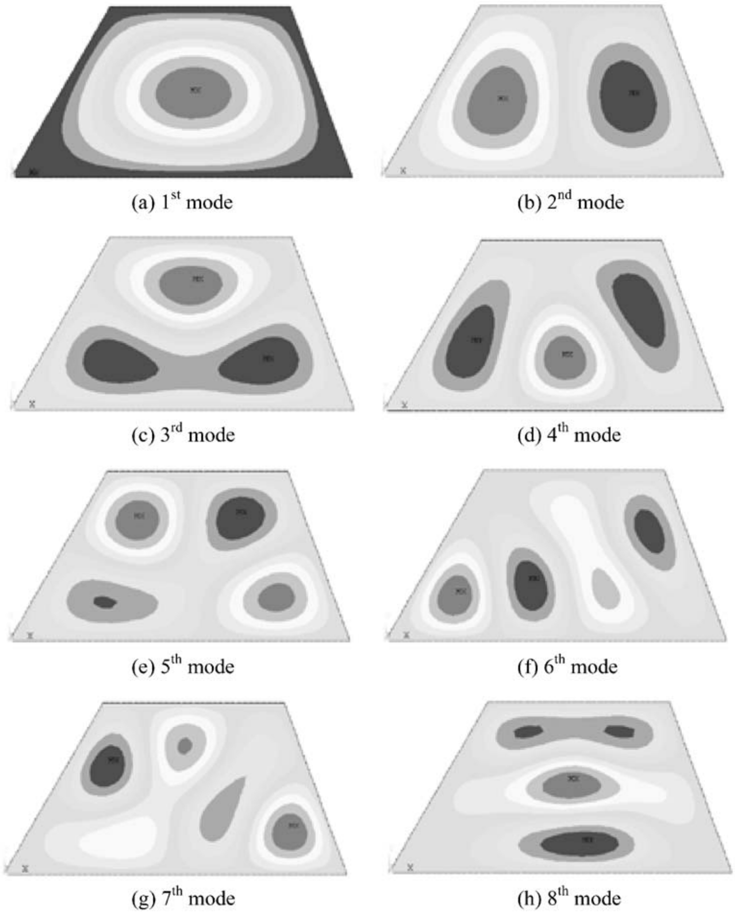

Fig. 8. First eight mode shapes of the trapezoidal membrane given by FEM (ANSYS) when 1447 nodes are used.

membrane. Note that the superscript *rec* of $\mathbf{SM}_{2N}^{rec}(k)$ is added to discriminate the system matrix of the rectangular membrane from that of the trapezoidal membrane.

The singular values of $\mathbf{SM}_{2N}^{rec}(k)$ are obtained by plotting logarithm values of $\det[\mathbf{SM}_{2N}^{rec}(k)]$ as shown in Fig. 4. The singular values of $\mathbf{SM}_{2N}^{rec}(k)$ are summarized in the 4th column of Table 1, which shows that they exactly coincide with the *incorrect singular values* of the trapezoidal membrane. This interesting fact can be used to remove unwanted troughs corresponding to the *incorrect singular values* in Fig. 3. To remove the unwanted troughs, logarithm values of $\det[\mathbf{SM}_{2N}(k)]/\det[\mathbf{SM}_{2N}^{rec}(k)]$ are plotted in Fig. 5, which shows that only 8 troughs ($\Lambda_1 - \Lambda_8$) corresponding to the *correct singular values* (the eigenvalues of the trapezoidal membrane) have been generated. The unwanted troughs disappear or are changed into crests in Fig. 5 is because the singularity of $\mathbf{SM}_{2N}^{rec}(k)$ is similar to or higher than that of $\mathbf{SM}_{2N}(k)$, respectively.

On the other hand, the exact eigenvalues of the rectangular membrane are compared with the singular values of $\mathbf{SM}_{2N}^{rec}(k)$ in Table 1 (compare the 4th column with 5th column in Table 1) to confirm whether all the singular values of $\mathbf{SM}_{2N}^{rec}(k)$ correspond to the eigenvalues of the rectangular membrane. This comparison indicates that only a part of the singular values coincides with the eigenvalues of the rectangular membrane. The other singular values are termed *the cut-off frequency values*, which denote frequency parameter values satisfying $k_x^{(\mathrm{I})} = 0$ or $k_x^{(\mathrm{II})} = 0$ in Eqs. (4) and (5).

3.3. Convergence of eigenvalues, and mode shapes

Fig. 6 shows the logarithm values of $\det[\mathbf{SM}_{2N}(k)]/\det[\mathbf{SM}_{2N}^{rec}(k)]$ for $N = 4, 5$, and 6 (logarithm values for $N = 3$ have been already shown in Fig. 5). A comparison between the proposed method and the numerical method is summarized in Table 2. In Table 2, the eigenvalues by the numerical method do not converge easily but the eigenvalues by the proposed method have already converged even for $N = 5$. This result confirms that the proposed method has excellent convergence characteristics. Shown in Fig. 7, the mode shapes of the trapezoidal membrane obtained by the proposed method agree well with those by the numerical method (see Fig. 8).

4. Conclusions

An effective approach suitable for the free vibration analysis of unsymmetric trapezoidal membranes was introduced in the paper. The example shows that eigenvalues generated by the proposed method rapidly converge whereas those by the numerical method (FEM) do not, and that the proposed method successfully gives accurate mode shapes.

Acknowledgements

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